

ANNOUNCEMENT: ON THE RATIONALIZATION OF THE $K(n)$ -LOCAL DETERMINANT SPHERE

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Throughout, let p be a prime, n some height, and denote the category of $K(n)$ -local spectra by $\mathrm{Sp}_{K(n)}$, equipped with the localized symmetric monoidal structure. The overarching goal of this project is to fully understand the rational homotopy of all invertible objects in $\mathrm{Sp}_{K(n)}$ using methods from p -adic geometry.

The algebraic Picard group of $\mathrm{Sp}_{K(n)}$ has recently been computed in [BSSW25a]: It is shown to be of rank two over $\mathbb{Z}_p$ for all $n \geq 2$, topologically generated by $\Sigma \mathbb{S}_{K(n)}$ and the determinant sphere $\mathbb{S}_{K(n)}[\det]$ at odd primes. Integer suspensions only shift homotopy groups; the determinant spheres are the next natural cases to compute rationally.

Theorem. *If $n \geq 2$ and $k \in \mathbb{Z}$, we have $\mathbb{S}_{K(n)}[\det^{\otimes k}] \otimes \mathbb{Q} = 0$ unless $k \in \{0, 1\}$.*

We recall that, for $k = 0$, the rational homotopy groups were computed by Barthel–Schlank–Stapleton–Weinstein [BSSW25b],

$$\pi_* \mathbb{S}_{K(n)} \otimes \mathbb{Q} \cong H_c^{-*}(\mathrm{GL}_n(\mathbb{Z}_p); \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \dots, x_n), \quad |x_i| = 1 - 2i,$$

so it remains to determine $\pi_* \mathbb{S}_{K(n)}[\det] \otimes \mathbb{Q}$. The main result of our work in progress carries out this computation at all primes and all heights:

Pre-Theorem. *We compute $\pi_* \mathbb{S}_{K(n)}[\det] \otimes \mathbb{Q}$ as a graded module over $\pi_* \mathbb{S}_{K(n)} \otimes \mathbb{Q}$. The Poincaré series of $\mathbb{S}_{K(n)}[\det]$ is*

$$\sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}_p}(\pi_k \mathbb{S}_{K(n)}[\det] \otimes \mathbb{Q}) \cdot T^k = T^{n-n^2} (L_n(T) + T \cdot L_{n-1}(T)),$$

with $L_0(T) = 1$ and for $n \geq 1$

$$(L) \quad L_n(T) = T^{2n} \cdot \sum_{\lambda \models n} \left(\prod_{i=1}^{\#\lambda} \varepsilon_{\lambda_i-1}(T) \right), \quad \varepsilon_m(T) = \prod_{j=1}^m (1 + T^{2j-1}),$$

where the sum in $L_n(T)$ ranges over the ordered partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n .

Via Gross–Hopkins duality [HG94a, Str00], the above results allow us to deduce the following qualitative and quantitative consequences for the (integral) homotopy groups of the L_n -local sphere:

Corollary. *For every prime p and height $n \geq 1$:*

- (a) *The homotopy group $\pi_k L_n \mathbb{S}$ is a direct sum of one copy of $\mathbb{Z}_{(p)}$ in $k = 0$, finitely many copies of $\mathbb{Z}/p^\infty$ when $k < 0$, and a bounded p -power torsion abelian group in each degree k .*
- (b) *Under the hypothesis that the canonical map*

$$(H) \quad \mathrm{Hom}(\mathbb{Q}, L_{n-1} \mathbb{S}) \rightarrow \mathrm{Hom}(\mathbb{Q}, L_{n-1} L_{K(n)} \mathbb{S})$$

is injective on homotopy groups in every degree, the locations of divisible subgroups in $\pi_* L_n \mathbb{S}$ are given by the polynomial $L_n(T)$ in (L):

$$L_n(T) = \sum_{k \geq 0} (\# \text{ of } \mathbb{Z}/p^\infty \text{ in } \pi_{-k} L_n \mathbb{S}) \cdot T^k.$$

We prove Hypothesis (H) at all primes for all heights $n \leq 3$. In particular, it is unaffected by the recent announced counterexample to the weak chromatic splitting conjecture [Ope26a, Ope26b]. In fact, combining our results with recent claims by OpenAI [Ope26c] would allow us to verify Hypothesis (H) also for all $n \geq 4$ and $p > n + 1$. Together with another OpenAI preprint [Ope26d], our main results would imply that, in the same generic range, the bounded torsion groups appearing in $\pi_* L_n S$ are finite. Forthcoming work of Barthel–Li–Yang–Zhang will report on the failure of the weak chromatic splitting conjecture for all primes p and all heights $n \geq 3$. We therefore view Hypothesis (H) as a suitable replacement sufficient to capture the structure of $\pi_* L_n S$.

Remark. *Let us add two historical comments.*

- (a) *The case $n = 2$ and $0 < k < p - 1$ of our main results has also been established in unpublished work of Hans-Werner Henn.*
- (b) *Previous work of Hovey [Hov95] and Morava [Mor14] also determines the polynomial $L_n(T)$ in (L) that records the location of the divisible summands in $\pi_* L_n S$, assuming the strong chromatic splitting conjecture. However, the latter has been claimed to be false by OpenAI [Ope26a].*

We finish this announcement with a brief outline of the key ingredients that go into the proofs of our main theorems. The key input is the Lubin–Tate and Drinfeld two-tower isomorphism in [SW20, SW13]. This isomorphism translates the cohomology on the Lubin–Tate side to the more accessible Drinfeld side:

$$(T) \quad [(LT_{n,\eta})^\diamond // \mathbb{G}_n] \simeq [(\mathcal{H}_{\mathbb{Q}_p}^{n-1})^\diamond // GL_n(\mathbb{Z}_p)].$$

Here $\mathbb{G}_n$ is the Morava stabilizer group and $\mathcal{H}^{n-1}$ is the Drinfeld symmetric space.

The first main step is to extend the methods in [BSSW25b] to integer powers of the determinant twist. To adapt the setup, we need to control how line bundles move across the two-tower isomorphism (T). Then we use integral p -adic Hodge theory [BMS18, ČK19], the Gross–Hopkins and the Hodge–Tate period maps [HG94b, SW13], and Colmez–Dospinescu–Nizioł’s computation of sheaf cohomology over the Drinfeld space in [CDN20], to reduce the rational determinant spheres to the continuous Steinberg representation St_n .

We have a proof of the following result, the details of which are currently being written down:

Pre-Theorem. *For every prime p and every height $n \geq 2$, we have an isomorphism*

$$\pi_* S_{K(n)}[\det] \otimes \mathbb{Q} \cong H_c^{2n-*}(GL_n(\mathbb{Z}_p); St_n^\vee \otimes \mathbb{Q}),$$

where $St_n^\vee$ is the weak continuous dual of St_n .

We view this as an unexpected connection between rational chromatic computations and continuous Steinberg representations. Finally, using Poincaré duality, we are reduced to computing the rational group cohomology of the relevant Steinberg representations. The latter can be calculated using algebraic discrete Morse theory [JW09].

Theorem. *The Poincaré series of the group cohomology $H_c^*(GL_n(\mathbb{Z}_p); St_n \otimes \mathbb{Q})$ is*

$$\sum_{k \geq 0} \dim_{\mathbb{Q}_p} [H_c^k(GL_n(\mathbb{Z}_p); St_n \otimes \mathbb{Q})] T^k = T^{-n}(L_n(T) + T \cdot L_{n-1}(T)),$$

for any $n \geq 1$.

This is one of the novel ingredients in our work in comparison to [BSSW25b], where the corresponding coefficients were simply the trivial module $\mathbb{Q}_p$.

Acknowledgements. TB and NZ are grateful to the Max Planck Institute for Mathematics for its hospitality. TB is supported by the ERC under Horizon Europe (grant No. 101042990). GL is partially supported by NSFC-12526206. NZ is partially supported by the NSF Grant DMS-2348963.

AI-declaration. Part of this work is based on extended interactions with AI models; our forthcoming write-up will contain a more thorough discussion.

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